

Two-Particle Correlations in the Wave Function and Covariant Current Approaches

Dmitry Anchishkin^{1,*} and Ulrich Heinz^{2,†}

¹ *Bogolyubov Institute for Theoretical Physics, 03068 Kiev, Ukraine*

² *Physics Department, The Ohio State University, Columbus, OH 43210*

(Dated: February 2, 2008)

We consider two-particle correlations, which appear in relativistic nuclear collisions due to the quantum statistics of identical particles, in the frame of two formalisms: wave-function and current. The first one is based on solution of the Cauchy problem, whereas the second one is a so-called current parametrization of the source of secondary particles. We argue that these two parameterizations of the source coincide when the wave function at freeze-out times is put in a specific correspondence with a current. Then, the single-particle Wigner density evaluated in both approaches gives the same result.

I. INTRODUCTION

The models and approaches which are used to describe the processes occurred in the reaction region in relativistic heavy-ion collisions are examined by comparison of provided predictions with experimental data on single-, two- and many-particle momentum spectra, which contain information about the source at the early stage (photons, dileptons) and at the stage of so called “freeze-out” (hadron spectra). Two-particle correlations or the Hanbury-Brown-Twiss interferometry (HBT) encapsulate information about the space-time structure and dynamics of the emitting source [1, 2, 3, 4, 5, 6, 7]. Usually, consideration of the correlations, which occur in relativistic heavy-ion collisions, assumes that: (i) the particles are emitted independently (or the source is completely chaotic), and (ii) finite multiplicity corrections can be neglected. Both approximations are expected to be good

*anch@bitp.kiev.ua

†heinz@mps.ohio-state.edu

for high energy nuclear collisions with large multiplicities. Then, correlations reflect a) the effects from symmetrization (antisymmetrization) of the amplitude to detect identical particles with certain momenta, and b) the effects which are generated by the final state interactions of the detected particles between themselves and with the source. On the first sight one can regard the final state interactions (FSI) as a contamination of “pure” particle correlations. But, it should be noted that the FSI depend on the structure of the emitting source and thus provide as well information about source dynamics [8].

Several surprising questions motivated by new experimental data appeared recently in the HBT. For instance, the experimental measurements on two-pion correlations [9, 10, 11, 12] give the ratio of $R_{\text{out}}/R_{\text{side}} \approx 1$, what is much smaller than that predicted theoretically (the so called “RHIC HBT Puzzle”). This raises the question to what extent some of the model predictions are consistent with experimental measurements [13, 14] or may be the observed discrepancies are due to such an “apples-with-oranges” comparison. All this drew attention and inspired a more detailed discussion of the theoretical background of the HBT. In the present paper we are going along this line, we would like to clarify a question concerning different kinds of parametrization exploited in the HBT.

The nominal quantity expressing the correlation function in terms of experimental distributions [2] is

$$C(\mathbf{k}_a, \mathbf{k}_b) = \frac{P_2(\mathbf{k}_a, \mathbf{k}_b)}{P_1(\mathbf{k}_a) P_1(\mathbf{k}_b)}, \quad (1)$$

where $P_1(\mathbf{k}) = E d^3 N / d^3 k$ and $P_2(\mathbf{k}_a, \mathbf{k}_b) = E_a E_b d^6 N / (d^3 k_a d^3 k_b)$ are single- and two-particle cross-sections.

In the absence of the final state interactions the theoretical expression for the two-particle correlator reads

$$C(q, K) = 1 \pm \frac{\left| \int d^4 X e^{iq \cdot X} S(X, K) \right|^2}{\int d^4 X S\left(X, K + \frac{q}{2}\right) \int d^4 Y S\left(Y, K - \frac{q}{2}\right)}, \quad (2)$$

where $K = (k_a + k_b)/2$, $q = k_a - k_b$. This expression was obtained in the different approaches. In the so called ”wave-function” approach [8] source function $S(X, K)$ is defined in the following way

$$S_{\text{wf}}(X, K) = \int d^4 x e^{iK \cdot x} \sum_{\gamma, \gamma'} \rho_{\gamma\gamma'} \psi_\gamma\left(X + \frac{x}{2}\right) \psi_{\gamma'}^*\left(X - \frac{x}{2}\right), \quad (3)$$

where $\rho_{\gamma\gamma'}$ is the density matrix which in thermal equilibrium has the form $\rho_{\gamma\gamma'} = \delta_{\gamma\gamma'} \exp(-E_\gamma/T)$. The wave function, $\psi_\gamma(t, \mathbf{x})$, is taken at freeze-out times, i.e. $t \in t_\Sigma$. Freeze-out hyper-surface Σ is a spatial surface which moves in space in the same way as, for instance, the surface of the balloon during pumping. It represents an imaginary border between two domains: inside the surface a strong dynamics takes place whereas outside the surface the particles propagate outward freely. Wave function at freeze-out times can be regarded as initial one for its further history and because its further evolution is free (we do not discuss final state interactions so far) it can be easily taken into account. As it intuitively understood the free evolution can be reverse back and resulting cross-section and other measurable physical quantities, for instance source function $S(X, K)$, are determined through initial values of the wave function, i.e. by the values of the wave function at freeze-out times. Rigorous evaluations give exactly this result. On the other hand, the strong dynamics which acts inside freeze-out hyper-surface results in creation of the quantum state at freeze-out times. Hence, the wave function at freeze-out times is a final state of the strong dynamics. Representing experimentally measured quantities with the help of these states we can study strong interactions in dense and hot nuclear matter. Because of this creativity the separation of the interaction scales in space and time which is made with the help of freeze-out hyper-surface looks so attractive.

Correlation function (2) was derived first in the model where essential point is a parametrization of the source by use of the currents $J_\gamma(x)$ [1] (see also [15]) which become then the constituent elements of the source function

$$S_{\text{cur}}(X, K) = \int d^4x e^{iK \cdot x} \sum_{\gamma, \gamma'} \rho_{\gamma\gamma'} J_\gamma \left(X + \frac{x}{2} \right) J_{\gamma'}^* \left(X - \frac{x}{2} \right). \quad (4)$$

As a matter of fact, both approaches should give the same result in the region where they are valid. The goal of this paper is to find relation between source functions (3) and (4) obtained in wave function approach and covariant current approach respectively.

II. SINGLE- AND TWO-PARTICLE CROSS SECTIONS WITHOUT FSI

In this section we consider the two-particle quantum statistical correlations when one neglects the final state interactions of the detected particles. This phenomenon is visualized most transparently on the bases of the standard quantum mechanics. First, we briefly con-

sider the so called wave function parametrization of the source in nonrelativistic approach. This approach allows one to include also into consideration the final state interactions [8]. Relativistic picture is considered on the base of the current parametrization and then on the base of the wave function parametrization of the source. First, we compare these two approaches in a non-relativistic sector and put in correspondence the source functions (3) and (4). After that the same comparison is carried out for relativistic sector.

A. Wave function parametrization of the source. Nonrelativistic approach

The probability to register two-particles which are created in the relativistic heavy ion collisions and have definite asymptotic momenta \mathbf{k}_a and \mathbf{k}_b is compared usually with the probability to register independently two particles with the same momenta. That is why, we first turn to consideration of the single-particle spectrum.

Let us consider a single-particle state ψ_γ emitted by the source. Its propagation to the detector is governed by the Schrödinger equation

$$i \frac{\partial \psi_\gamma(\mathbf{x}, t)}{\partial t} = \hat{h}(\mathbf{x}) \psi_\gamma(\mathbf{x}, t), \quad (5)$$

where $\hat{h}(\mathbf{x}) = -\frac{1}{2m}\nabla^2$. The index γ denotes a complete set of 1-particle quantum numbers. Equation (5) is solved by $\psi_\gamma(\mathbf{x}, t, t_0) = \exp[-i\hat{h}(\mathbf{x})(t - t_0)] \psi_\gamma(\mathbf{x}, t_0)$ in terms of the single-particle wave function at some initial time t_0 , see Fig.1. For the spherically symmetric fireball the values of the wave function $\psi_\gamma(\mathbf{x}, t_0)$ parameterize the “freeze-out distribution” of the particles inside the sphere of the radius R_1 as it is depicted in Fig.1. We assume that the detector measures asymptotic momentum eigenstates, i.e. that it acts by projecting the emitted single-particle state onto $\phi_{\mathbf{k}}^{\text{out}}(\mathbf{x}, t) = \exp[i\mathbf{k} \cdot \mathbf{x} - i\omega(\mathbf{k})t]$, where $\omega(\mathbf{k}) = \mathbf{k}^2/2m$. The measured single-particle momentum amplitude is then

$$A_\gamma(\mathbf{k}, t_0) = \lim_{t \rightarrow \infty} \int d^3x \phi_{\mathbf{k}}^{\text{out},*}(\mathbf{x}, t) \psi_\gamma(\mathbf{x}, t, t_0). \quad (6)$$

The single-particle probability to detect the particle with certain momentum is obtained by averaging (6) and its complex conjugate with the density matrix $\rho_{\gamma\gamma'}$ defining the source dynamics. This density matrix is characterized by a probability distribution for the single-particle quantum numbers γ and by a distribution of emission times t_0 . We write

$$P_1(\mathbf{k}) = \sum_{\gamma, \gamma'} \rho_{\gamma\gamma'} A_\gamma(\mathbf{k}, t_0) A_\gamma^*(\mathbf{k}, t_0). \quad (7)$$

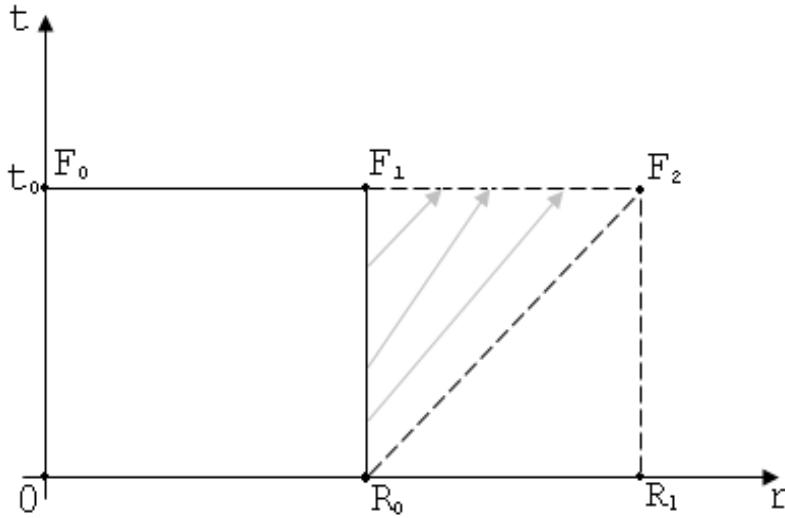


Figure 1: Sudden freeze-out hyper-surface $F_0F_1R_0$ for spherically symmetric fireball.

We define the single particle Wigner density of the source which with accounting for emission times, $t = t_0$, reads

$$S\left(\frac{x_1 + x_2}{2}, K\right) = \int d^4(x_1 - x_2) \delta(x_1^0 - t_0) \delta(x_2^0 - t_0) e^{iK \cdot (x_1 - x_2)} \sum_{\gamma, \gamma'} \rho_{\gamma\gamma'} \psi_\gamma(x_1) \psi_{\gamma'}^*(x_2). \quad (8)$$

This function accumulates all information about the source which emits the particles. Making transformation to new coordinates, $X = (x_1 + x_2)/2$, $x = x_1 - x_2$, the source function gets the form

$$S(X, K) = \delta(X_0 - t_0) \int d^4x \delta(x_0) e^{iK \cdot x} \sum_{\gamma, \gamma'} \rho_{\gamma\gamma'} \psi_\gamma\left(X + \frac{x}{2}\right) \psi_{\gamma'}^*\left(X - \frac{x}{2}\right). \quad (9)$$

Then, the expression for the single-particle spectrum (7) can be rewritten with making use of the source function

$$P_1(\mathbf{K}) = \int d^4X S(X, K). \quad (10)$$

Note, the factor $\delta(X_0 - t_0)\delta(x_0)$ in (9) carries information about space-like hypersurface where initial values of the wave function are given. For the sake of simplicity of the general scheme we start our consideration from a flat hypersurface, $t = t_0$, depicted in Fig.1. We turn to an arbitrary hypersurface in the next sections where relativistic approach is elaborated.

Let us consider a two-particle state ψ_γ emitted by the source. Its propagation to the

detector is governed by the Schrödinger equation

$$i \frac{\partial \psi_\gamma(\mathbf{x}_a, \mathbf{x}_b, t)}{\partial t} = \hat{H}(\mathbf{x}_a, \mathbf{x}_b) \psi_\gamma(\mathbf{x}_a, \mathbf{x}_b, t), \quad (11)$$

where $\hat{H}(\mathbf{x}_a, \mathbf{x}_b) = \hat{h}(\mathbf{x}_a) + \hat{h}(\mathbf{x}_b)$. The index γ denotes a complete set of 2-particle quantum numbers. Equation (11) is solved by

$$\psi_\gamma(\mathbf{x}_a, \mathbf{x}_b, t) = \exp[-i\hat{H}(\mathbf{x}_a, \mathbf{x}_b)(t - t_0)] \psi_\gamma(\mathbf{x}_a, \mathbf{x}_b, t_0), \quad (12)$$

in terms of the two-particle wave function at some initial time t_0 . Detector acts by projecting the emitted two-particle state onto $\phi_{\mathbf{p}_a, \mathbf{p}_b}^{\text{out}}(\mathbf{x}_a, \mathbf{x}_b, t) = e^{i(\mathbf{p}_a \cdot \mathbf{x}_a - \omega_a t)} e^{i(\mathbf{p}_b \cdot \mathbf{x}_b - \omega_b t)}$, where $\omega_{a,b} = \mathbf{p}_{a,b}^2/2m^2$. We will only consider the case of pairs of identical particles, $m_a = m_b = m$. The measured two-particle momentum amplitude is then

$$A_\gamma(\mathbf{k}_a, \mathbf{k}_b) = \lim_{t \rightarrow \infty} \int d^3x_a d^3x_b \phi_{\mathbf{k}_a, \mathbf{k}_b}^{\text{out},*}(\mathbf{x}_a, \mathbf{x}_b, t) \psi_\gamma(\mathbf{x}_a, \mathbf{x}_b, t). \quad (13)$$

We assume the two particles are emitted independently, implying that at some freeze-out time t_a the two-particle wave function $\psi_\gamma(\mathbf{x}_a, \mathbf{x}_b, t)$ factorizes

$$\psi_\gamma(\mathbf{x}_a, \mathbf{x}_b, t_a) = \frac{1}{\sqrt{2}} [\psi_{\gamma_a}(\mathbf{x}_a, t_a) \psi_{\gamma_b}(\mathbf{x}_b, t_a) \pm \psi_{\gamma_a}(\mathbf{x}_b, t_a) \psi_{\gamma_b}(\mathbf{x}_a, t_a)]. \quad (14)$$

The indices γ_a, γ_b on the single-particle wave functions now label complete sets of single-particle quantum numbers. The time moment $t_a = t_0$ is the emission time of the latest emitted particle. Because of the symmetry of the wave function (14) it does not matter what time is nominated as latest one, t_a or t_b . By this we assume that symmetrization occurs when the last of the two particles is frozen out from a strongly interacting bulk.

After hermitian inversion of the evolution operator and applying it to symmetrised (antisymmetrised) out-state two-particle amplitude (13) gets the form

$$A_{\gamma_a, \gamma_b}(\mathbf{k}_a, \mathbf{k}_b, t_0) = \frac{1}{\sqrt{2}} \int d^3x_a d^3x_b [e^{-i(k_a \cdot x_a + k_b \cdot x_b)} \pm e^{-i(k_a \cdot x_b + k_b \cdot x_a)}]^* \psi_{\gamma_a}(x_a) \psi_{\gamma_b}(x_b), \quad (15)$$

where $x_a^0 = t_0$ and $x_b^0 = t_0$ and by relabeling the variables of integration we transferred symmetrization from the state (14) onto out-state. By this we represent the measured two-particle momentum amplitude as projection of non-symmetrized two-particle wave function taking at emission times onto symmetrised (antisymmetrised) plane waves taking as well at emission times.

The two-particle probability to detect two particles with momenta \mathbf{k}_a and \mathbf{k}_b is obtained by averaging two-particle amplitude (15) and its complex conjugate with the density matrix defining the source. This density matrix is characterized by a probability distribution for the two-particle quantum numbers (γ_a, γ_b) , also we average by a distribution of emission times (t_a, t_b) . We write

$$P_2(\mathbf{k}_a, \mathbf{k}_b) = \sum_{\gamma_a \gamma_b, \gamma_{a'} \gamma_{b'}} \rho_{\gamma_a \gamma_{a'}} \rho_{\gamma_b \gamma_{b'}} A_{\gamma_a' \gamma_{b'}}(\mathbf{k}_a, \mathbf{k}_b; t_0) A_{\gamma_a \gamma_b}^*(\mathbf{k}_a, \mathbf{k}_b; t_0). \quad (16)$$

We made the ansatz $\rho_{\gamma_a \gamma_b, \gamma_{a'} \gamma_{b'}} = \rho_{\gamma_a \gamma_{a'}} \rho_{\gamma_b \gamma_{b'}}$ which factorizes initial density matrix $\rho_{\gamma_a \gamma_b, \gamma_{a'} \gamma_{b'}}$ in such a way that independent emission of the two particles is ensured.

After straightforward algebra we write expression for the two-particle probability

$$\begin{aligned} P_2(q, K) = & \int d^4 X S\left(X, K + \frac{q}{2}\right) \int d^4 Y S\left(Y, K - \frac{q}{2}\right) \pm \\ & \pm \int d^4 X e^{iq \cdot X} S(X, K) \int d^4 Y e^{-iq \cdot Y} S(Y, K). \end{aligned} \quad (17)$$

Finally we get the two-particle correlator $C(q, K)$ (2), as a ratio of two-particle probability (17) and single-particle probabilities (10), where the source function $S(X, K)$ is defined in accordance with Eq. (9), i.e. all integrations are taken at emission times or on the freeze-out hyper-surface.

B. Current parametrization of the source

Let us consider a single-particle state Ψ_γ emitted by the source which we parametrized by the "source current" $J_\gamma(x)$. Its propagation to the detector is governed by the Klein-Gordon equation

$$(\partial_\mu \partial^\mu + m^2) \Psi_\gamma(x) = J_\gamma(x), \quad (18)$$

where $\partial_\mu \partial^\mu = \partial_t^2 - \vec{\nabla}^2$. The index γ denotes a complete set of 1-particle quantum numbers. (In a basis of wave packets these could contain the centers \mathbf{X} of the wave packets of the particles at their freeze-out times t .)

Single-particle momentum amplitude is defined as projection of the wave function $\Psi_\gamma(x)$ at "detector time $t = x^0$ " on the out-state $\phi_{\mathbf{k}}^{\text{out}}$,

$$A_\gamma(\mathbf{k}) = \lim_{x^0 \rightarrow \infty} (\phi_{\mathbf{k}}^{\text{out},*}(x), \Psi_\gamma(x)) = \lim_{t \rightarrow \infty} \int d^3 x \phi_{\mathbf{k}}^{\text{out},*}(\mathbf{x}, t) i \overleftrightarrow{\partial}_t \Psi_\gamma(\mathbf{x}, t), \quad (19)$$

where by definition $a(t) \stackrel{\leftrightarrow}{\partial}_t b(t) \equiv a\partial_t b - (\partial_t a)b$. We assume that the detector measures asymptotic momentum eigenstates, i.e. that it acts by projecting the emitted single-particle state onto

$$\phi_{\mathbf{k}}^{\text{out}}(\mathbf{x}, t) = e^{-i\omega(\mathbf{k})t + i\mathbf{k}\cdot\mathbf{x}} \equiv f_k^{(+)}(x), \quad (20)$$

where $\omega(\mathbf{k}) = \sqrt{m^2 + \mathbf{k}^2}$. Then, momentum amplitude can be rewritten as

$$A_\gamma(\mathbf{k}) = \lim_{x^0 \rightarrow \infty} \int d^3x \int d^4y f_k^{(+),*}(x) i \stackrel{\leftrightarrow}{\partial}_{x^0} G_R(x - y) J_\gamma(y). \quad (21)$$

Substituting to (21) the Green's function $(f_k^{(-)}(x) \equiv e^{i\omega(\mathbf{k})t - i\mathbf{k}\cdot\mathbf{x}})$

$$G_R(x - y) = i \theta(x^0 - y^0) \int \frac{d^3k}{(2\pi)^3 2\omega(\mathbf{k})} [f_k^{(+)}(x) f_k^{(+),*}(y) - f_k^{(-)}(x) f_k^{(-),*}(y)],$$

which satisfies equation $(\partial_\mu \partial^\mu + m^2)G_R(x - y) = \delta^4(x - y)$, and using orthogonal properties of the basic functions, $\int d^3x f_{k_a}^{(\pm),*}(x) i \stackrel{\leftrightarrow}{\partial}_{x^0} f_{k_b}^{(\pm)}(x) = \pm(2\pi)^3 2\omega(\mathbf{k}_a) \delta^3(\mathbf{k}_a - \mathbf{k}_b)$, and $\int d^3x f_{k_a}^{(+),*}(x) i \stackrel{\leftrightarrow}{\partial}_{x^0} f_{k_b}^{(-)}(x) = 0$, after straightforward calculation we come to the answer

$$A_\gamma(\mathbf{k}) = i \int d^4y e^{i\omega(\mathbf{k})y^0 - \mathbf{k}\cdot\mathbf{y}} J_\gamma(y), \quad (22)$$

where integration is taken over infinite space-time volume and that is why the finiteness in space and time of the particle source which we deal with is accumulated in the "cut-function" $J_\gamma(y)$. Moreover, it should be pointed out that amplitude $A_\gamma(\mathbf{k})$ in (22) is nothing more as the on-shell Fourier transformation of the current, hence in this approach the amplitude to register the particle with certain momentum \mathbf{k} directly reflects the model of the source which is settled by the particular definition of the current $J_\gamma(x)$.

The single-particle probability is obtained by averaging (22) and its complex conjugate with the density matrix defining the source. This density matrix is characterized by a probability distribution for the single-particle quantum numbers γ . We write

$$P_1(\mathbf{p}) = \sum_{\gamma, \gamma'} \rho_{\gamma\gamma'} A_\gamma(\mathbf{p}) A_{\gamma'}^*(\mathbf{p}), \quad (23)$$

Inserting (22) into (23) and using definition of the source function (4) after simple algebra we come to result

$$P_1(\mathbf{k}) = \int d^4x S(x, k), \quad (24)$$

which gives the single-particle probability in the same form as we obtained in Eq. (10) for the wave function approach but in contrast to the wave function approach the integration in (24) is taken over infinite space-time volume.

Two-particle momentum amplitude is defined as projection of the symmetrized (antisymmetrized) two-particle wave function $\psi_\gamma(x_a, x_b)$ at "detector times $x_a^0 \rightarrow \infty$ and $x_b^0 \rightarrow \infty$ " where the index γ denotes a complete set of two-particle quantum numbers, on the momentum eigen state $\phi_{\mathbf{k}_a, \mathbf{k}_b}^{\text{out}}$,

$$\begin{aligned} A_\gamma(\mathbf{k}_a, \mathbf{k}_b) &= \lim_{x_a^0 \rightarrow \infty} \lim_{x_b^0 \rightarrow \infty} (\phi_{\mathbf{k}_a, \mathbf{k}_b}^{\text{out},*}(x_a, x_b), \Psi_\gamma(x_a, x_b)) \\ &= \lim_{x_a^0 \rightarrow \infty} \lim_{x_b^0 \rightarrow \infty} \int d^3 x_a d^3 x_b \phi_{\mathbf{k}_a, \mathbf{k}_b}^{\text{out},*}(x_a, x_b) i \overleftrightarrow{\partial}_{x_a^0} i \overleftrightarrow{\partial}_{x_b^0} \Psi_\gamma(x_a, x_b), \end{aligned} \quad (25)$$

We label out-state by the values of measured momenta, i.e. \mathbf{k}_a and \mathbf{k}_b . The out-state at detector times $x_a^0 \rightarrow \infty$, $x_b^0 \rightarrow \infty$ reads

$$\phi_{\mathbf{k}_a, \mathbf{k}_b}^{\text{out}}(x_a, x_b) = f_{k_a}^{(+)}(x_a) f_{k_b}^{(+)}(x_b). \quad (26)$$

If the source is completely chaotic, i.e. the particles are emitted independently, that implies that the two-particle wave function is a product of two single-particle ones. For pairs of identical bosons (fermions) the two-particle wave function describing their propagation towards the detector must be symmetrized (anti-symmetrized). Taking the same arguments as for the wave function approach about delay of emission of one particle with respect to another one we write

$$\begin{aligned} \Psi_{\gamma_a, \gamma_b}(x_a, x_b) &= \frac{1}{\sqrt{2}} [\Psi_{\gamma_a}(x_a) \Psi_{\gamma_b}(x_b) \pm \Psi_{\gamma_a}(x_b) \Psi_{\gamma_b}(x_a)] = \\ &= \int d^4 y_a d^4 y_b G_R(x_a - y_a) G_R(x_b - y_b) \frac{1}{\sqrt{2}} [J_{\gamma_a}(y_a) J_{\gamma_b}(y_b) \pm J_{\gamma_a}(y_b) J_{\gamma_b}(y_a)] \\ &= \int \frac{d^3 k_a}{(2\pi)^3 2\omega(\mathbf{k}_a)} \frac{d^3 k_b}{(2\pi)^3 2\omega(\mathbf{k}_b)} f_{k_a}^{(+)}(x_a) f_{k_b}^{(+)}(x_b) \frac{1}{\sqrt{2}} [\tilde{J}_{\gamma_a}(k_a) \tilde{J}_{\gamma_b}(k_b) \pm \tilde{J}_{\gamma_a}(k_b) \tilde{J}_{\gamma_b}(k_a)], \end{aligned} \quad (27)$$

where we use solution of the Klein-Gordon equation (18) which is obtained with a help of the Green's function $G_R(x - y)$ and

$$\tilde{J}_\gamma(k) = \int d^4 y e^{i\omega(\mathbf{k})y^0 - \mathbf{k}\cdot\mathbf{y}} J_\gamma(y), \quad (28)$$

is the on-shell Fourier transformed source current. We do not write in (27) the negative-frequency piece of the Green's function because it evidently disappears on the next step: a projection of the wave function $\Psi_{\gamma_a, \gamma_b}(x_a, x_b)$ onto out-state. Indeed, to obtain the momentum amplitude we substitute expression (27) into (25) and use orthogonality relations of the basic functions $f_k^{(\pm)}(x)$. All this results in a simple final expression

$$A_{\gamma_a, \gamma_b}(\mathbf{k}_a, \mathbf{k}_b) = \frac{1}{\sqrt{2}} [\tilde{J}_{\gamma_a}(k_a) \tilde{J}_{\gamma_b}(k_b) \pm \tilde{J}_{\gamma_a}(k_b) \tilde{J}_{\gamma_b}(k_a)] \Big|_{k_a^0 = \omega(\mathbf{k}_a), k_b^0 = \omega(\mathbf{k}_b)}. \quad (29)$$

The two-particle probability is obtained by averaging amplitude (29) and its complex conjugate with the density matrix defining the source. This density matrix is characterized by a probability distribution for the two-particle quantum numbers (γ_a, γ_b)

$$P_2(\mathbf{k}_a, \mathbf{k}_b) = \sum_{\gamma_a \gamma_b, \gamma_{a'} \gamma_{b'}} \rho_{\gamma_a \gamma_b, \gamma_{a'} \gamma_{b'}} A_{\gamma_a \gamma_b}(\mathbf{k}_a, \mathbf{k}_b) A^*_{\gamma_{a'} \gamma_{b'}}(\mathbf{k}_a, \mathbf{k}_b), \quad (30)$$

As in the wave function approach we made the ansatz $\rho_{\gamma_a \gamma_b, \gamma_{a'} \gamma_{b'}} = \rho_{\gamma_a \gamma_{a'}} \rho_{\gamma_b \gamma_{b'}}$, which factorizes in such a way that independent emission of the two particles is ensured.

Substituting momentum amplitude (29) into (30), using definition (4) of the source function $S(Y, K)$, we can write for the two-particle probability

$$P_2(\mathbf{k}_a, \mathbf{k}_b) = \int d^4X S(X, k_a) \int d^4Y S(Y, k_b) \pm \int d^4X e^{iq \cdot X} S(X, K) \int d^4Y e^{-iq \cdot Y} S(Y, K), \quad (31)$$

which coincide with expression (17) obtained in wave function approach, consequently, we obtain correlator in the form (2). The integration on the right hand side of Eq. (31) is just taken over an infinite space-time interval, whereas in (17) the integration is taken over freeze-out hyper surface, or over initial times.

C. Wave function parametrization versus current parametrization. Nonrelativistic approach

The goal of this subsection is to put in correspondence the “wave function” approach which was elaborated in the Section II.A to the “current” approach of the Section II.B. To do this we consider this correspondence first in the non-relativistic limit and then fully relativistic comparing will be carried out.

We are going to obtain the current approach in the non-relativistic limit (see Appendix A). We make a standard unitary transformation of the wave function to extract oscillations associated with particle mass $\Psi_\gamma(x) = e^{-imx^0} \psi_\gamma(x)$. With respect to the new wave function $\psi_\gamma(x)$ the basic equation (18) reads

$$\left(i\partial_t + \frac{1}{2m} \nabla^2 \right) \psi_\gamma(t, \mathbf{x}) = j_\gamma(t, \mathbf{x}), \quad (32)$$

where

$$j_\gamma(t, \mathbf{x}) = -\frac{1}{2m} e^{imt} J_\gamma(t, \mathbf{x}). \quad (33)$$

and we skipped all terms of the order $1/c^2$ and higher, they serve as relativistic corrections. By this derivation we put in correspondence the current in the relativistic parametrization of the source with the current in the non-relativistic one.

With respect to the quantum state $\psi_\gamma(t, \mathbf{x})$ the momentum amplitude can be rewritten in the following way

$$A_\gamma(\mathbf{k}) = \lim_{x^0 \rightarrow \infty} \int d^3x \int d^4y f_k^{(+),*}(x) G_0(x - y) j_\gamma(y) = i \int d^4y e^{i\omega(\mathbf{k})y^0 - \mathbf{k} \cdot \mathbf{y}} j_\gamma(y) = i \tilde{j}_\gamma(k), \quad (34)$$

where

$$G_0(x - y) = - \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik \cdot (x-y)}}{k^0 - \omega(\mathbf{k}) + i\epsilon} = i \theta(x^0 - y^0) \int \frac{d^3k}{(2\pi)^3} e^{-ik \cdot (x-y)} \Big|_{k^0 = \omega(\mathbf{k})}, \quad (35)$$

is the Green's function which satisfies equation $(i\partial_t + \nabla^2/2m) G_0(x - y) = \delta^4(x - y)$ and $\tilde{j}_\gamma(k)$ is the on-shell Fourier transformation of the current.

Compare the amplitude (34), i.e. $A_\gamma(\mathbf{k}) = i \tilde{j}_\gamma(k)$, with the correspondent amplitude obtained in the relativistic case (22) we see that they coincide with one another, just in place of the capital letter J one should put a small one. Hence, the same transformation should be done in the definition of the source function (4).

Non-relativistic Schrödinger equation $(i\partial_t - \hat{H}(\mathbf{x})) \psi_\gamma(x) = 0$ supplemented by the initial condition, $\psi_\gamma(t = t_0, \mathbf{x}) = \Phi_\gamma(\mathbf{x})$, can be written, as was shown in Appendix B (the generalized Cauchy problem [16]), in the form of the Schrödinger equation with the source on the r.h.s. of equation which is defined at initial time $t = t_0$,

$$(i\partial_t - \hat{H}(\mathbf{x})) \psi_\gamma(t, \mathbf{x}) = i \Phi_\gamma(\mathbf{x}) \delta(t - t_0), \quad (36)$$

which is valid for $t \geq t_0$. Then, one can solve this equation with the help of the Green's function (35) and write solution in the following form

$$\begin{aligned} \psi_\gamma(t - t_0, \mathbf{x}) &= i \int d^3y G_0(t - t_0, \mathbf{x} - \mathbf{y}) \Phi_\gamma(\mathbf{y}) \\ &= \theta(t - t_0) \int \frac{d^3p}{(2\pi)^3} d^3x' e^{-i\omega(\mathbf{p})(t-t_0) + i\mathbf{p} \cdot (\mathbf{x}-\mathbf{x}')} \psi_\gamma(t_0, \mathbf{x}'). \end{aligned} \quad (37)$$

As a matter of fact, this solution coincides with that one obtained with the help of the evolution operator, $\psi_\gamma(t - t_0, \mathbf{x}) = \theta(t - t_0) e^{-i\hat{H}(\mathbf{p}, \mathbf{x})(t-t_0)} \psi_\gamma(t_0, \mathbf{x})$, which we exploited in the wave function approach in paragraph II A.

On the other hand, solving the Cauchy problem in this way one can consider the expression on the r.h.s. of eq.(36) as a specific current

$$j_\gamma(t, \mathbf{x}) = i \psi_\gamma(t, \mathbf{x}) \delta(t - t_0). \quad (38)$$

Then, going through all preceding steps to evaluate the source function $S(X, K)$ with making use of this specific current one evidently discovers that the source functions (4) and (9) coincide with one another. We regard this result as first example when two types of parametrization of the source can give the same answer for specific connection between current and wave function given at freeze-out times.

We are going now to prove that the same is valid in more general case. Indeed, let us write once more the solution of eq.(32)

$$\begin{aligned}\psi_\gamma(t, \mathbf{x}) &= \int d^4z G_0(t - z_0, \mathbf{x} - \mathbf{z}) j_\gamma(z_0, \mathbf{z}) \\ &= i \int d^4z d^3y G_0(t - t_0, \mathbf{x} - \mathbf{y}) G_0(t_0 - z_0, \mathbf{y} - \mathbf{z}) j_\gamma(z_0, \mathbf{z}),\end{aligned}\quad (39)$$

where in the second line we split the Green's function at the point $t = t_0$ using the group property of the Green's functions. We are making now the physical input: let us prepare the initial state, which will be used in the wave function parametrization of the source, $\Phi_\gamma(\mathbf{y}) = \psi_\gamma(t_0, \mathbf{y})$, in the following way (note, up to now we did not specialize a generation of the wave function at freeze-out times)

$$\Phi_\gamma(\mathbf{y}) = \int d^4z G_0(t_0 - z_0, \mathbf{y} - \mathbf{z}) j_\gamma(z_0, \mathbf{z}). \quad (40)$$

Then, rewriting the second line in (39) with making use of the state $\Phi_\gamma(\mathbf{y})$ (just defined in (40)) we obtain the single-particle quantum state, $\psi_\gamma(x)$, at the times which are after $t = t_0$, i.e. after freeze-out, in the following form

$$\psi_\gamma(t, \mathbf{x}) = i \int d^3y G_0(t - t_0, \mathbf{x} - \mathbf{y}) \psi_\gamma(t_0, \mathbf{y}). \quad (41)$$

What is most interesting, expression (41) is exactly a solution of the Schrödinger equation (36) with $\psi_\gamma(t_0, \mathbf{y})$ as the initial condition (see (37)).

Let us make one note. In eq.(39) after splitting of the Green's function we meet the product of two θ -functions, $\theta(t - t_0)\theta(t_0 - z_0)$. Because we are interesting in detector times the value of time t goes to infinity and we can avoid the first θ -function. At the same time the second θ -function cuts an action of the source current at the times $t = t_0$. But this feature does not distort the influence of the current if t_0 is the freeze-out time. In other words, we assume that a life time of the current coincides with a life time of the fireball, $j_\gamma(t, \mathbf{x}) \propto \theta(t_0 - t)$.

So, we obtain the same quantum state $\psi_\gamma(t, \mathbf{x})$ in two approaches: 1) The current parametrization of the source, expression (39), first line, which is solution of eq.(32); and 2) The wave function parametrization of the source, expression (41), which is solution of the Cauchy problem for the Schrödinger equation. If we start a description of the propagation of the particle to detector from the quantum state $\psi_\gamma(t, \mathbf{x})$ taken from (41) and go through all steps to the source function $S(X, K)$ we come to expression (9) which is evaluated with the help of the initial states $\psi_\gamma(t_0, \mathbf{y})$. In fact, as was shown in the section II A, by this we obtain the source function exploiting the wave function parametrization of the source. On the other hand, we can start the evaluation from the same quantum state $\psi_\gamma(t, \mathbf{x})$ taking it in the form (39) (first line). Then, we come to the source function (4) obtained in the current approach. Meanwhile, the starting point for both expressions is the same state $\psi_\gamma(t, \mathbf{x})$ (what gives the same amplitude, the same single-particle probability and so on). Hence, if the initial wave function, $\Phi_\gamma(\mathbf{x})$, and the current, $j_\gamma(x)$, are connected to one another by eq.(40), then both evaluations of the source function $S(X, K)$ give the same result. Thus, the single-particle Wigner functions constructed in both approaches are equal

$$\begin{aligned} S(Y, K) &= \delta(Y^0 - t_0) \int d^4y \delta(y^0) e^{iK \cdot y} \sum_{\gamma, \gamma'} \rho_{\gamma\gamma'} \psi_\gamma \left(Y + \frac{y}{2} \right) \psi_{\gamma'}^* \left(Y - \frac{y}{2} \right) \\ &= \int d^4y e^{iK \cdot y} \sum_{\gamma, \gamma'} \rho_{\gamma\gamma'} j_\gamma \left(Y + \frac{y}{2} \right) j_{\gamma'}^* \left(Y - \frac{y}{2} \right). \end{aligned} \quad (42)$$

Consequently, the single-particle spectrum and two-particle correlations taken in the wave function parametrization of the source coincide with the respective spectra taken in the current parametrization.

Connection (40) between current and wave function at freeze-out times has a transparent physical interpretation: the action of the current which describes in a semi-classical way a creation of secondary particles during the life time of the fireball can be accumulated in the wave function at freeze-out times. That is why, it does not matter what quantity is used then to describe the free propagation of the particles to detector. Moreover, the correspondence (41), as it is seen in Fig.1, results in extension of the effective volume where initial wave function is given by adding a spherical layer for the radii r in the limits $R_0 \leq r \leq R_1$. Indeed, all particles which were emitted during life time of the fireball from the boundary $F_1 R_0$ accumulated now on the space-like segment $F_1 F_2$. This means that the wave function $\psi_\gamma(t_0, \mathbf{x})$ given on the extended space-like hypersurface, segment $F_0 F_1 F_2$, takes into account all secondary particles which were “produced” by the source current.

D. Wave-function versus current parametrization of the source in relativistic approach

First, we consider the wave-function parametrization of the source. Single-particle momentum amplitude is evaluated as projection of the wave function, $\Psi_\gamma(t, \mathbf{x})$, taken at asymptotic times onto out-state $\phi_{\mathbf{p}}^{\text{out}}$

$$A_\gamma(\mathbf{p}) = \lim_{t \rightarrow \infty} \int d^3x \phi_{\mathbf{p}}^{\text{out},*}(t, \mathbf{x}) i \overleftrightarrow{\partial}_t \Psi_\gamma(t, \mathbf{x}), \quad (43)$$

where $\phi_{\mathbf{p}}^{\text{out}}(t, \mathbf{x}) = e^{-i\omega(\mathbf{p})t+i\mathbf{p}\cdot\mathbf{x}}$. The wave function, $\Psi_\gamma(t, \mathbf{x})$ is solution of the Klein-Gordon equation, $(\partial_\mu \partial^\mu + m^2)\Psi_\gamma(x) = 0$, which is supplemented by the initial conditions

$$\Psi_\gamma(t, \mathbf{x})|_{t=t_0} = \Phi_{\gamma 0}(\mathbf{x}), \quad \text{and} \quad \left. \frac{\partial \Psi_\gamma(t, \mathbf{x})}{\partial t} \right|_{t=t_0} = \Phi_{\gamma 1}(\mathbf{x}). \quad (44)$$

As we show in Appendix B (see (B17)) this problem can be formulated as equation with a source which constructed with a use of the initial conditions (44)

$$(\partial_\mu \partial^\mu + m^2)\Psi_\gamma(t, \mathbf{x}) = \Phi_{\gamma 1}(\mathbf{x}) \delta(t - t_0) + \Phi_{\gamma 0}(\mathbf{x}) \delta'(t - t_0), \quad (45)$$

which is valid for times $t \geq t_0$. Solving this equation with a help of the Green's function and inserting solution to (43) one can write the amplitude in the following form

$$A_\gamma(\mathbf{p}) = \lim_{x^0 \rightarrow \infty} \int d^3x \int d^4y \delta(y^0 - t_0) f_p^{(+)*}(x) i \overleftrightarrow{\partial}_{x^0} G_R(x - y) \overleftrightarrow{\partial}_{y^0} \Psi_\gamma(y), \quad (46)$$

where $f_p^{(+)}(x) = \phi_{\mathbf{p}}^{\text{out}}(t, \mathbf{x})$. Taking $G_R(x - y)$ in the explicit form and using the orthogonal properties of the basic functions $f_p^{(\pm)}(x)$ we come to the answer

$$A_\gamma(\mathbf{p}) = \int d^4y \delta(y^0 - t_0) \left[f_p^{(+)*}(y) i \overleftrightarrow{\partial}_{y^0} \Psi_\gamma(y) \right] = \Psi_\gamma^{(+)}(t_0, \mathbf{p}), \quad (47)$$

where $\Psi_\gamma^{(+)}(t_0, \mathbf{p})$ is the Fourier component of a positive-energy piece of the function $\Psi_\gamma(t_0, \mathbf{y})$. Note, the wave function consists from two contributions, positive- and negative-energy defined, $\Psi_\gamma(x) = \Psi_\gamma^{(+)}(x) + \Psi_\gamma^{(-)}(x)$, respectively. Actually, for the sake of simplicity we consider a flat space-like hypersurface, $t = t_0$, the segment $F_1 F_2 F_3$, as it is depicted in Fig.2. At the same time, the amplitude (47) can be expressed in the covariant form as well

$$A_\gamma(\mathbf{p}) = \int d\sigma^\mu(y) f_p^{(+)*}(y) i \overleftrightarrow{\partial}_{y^\mu} \Psi_\gamma(y). \quad (48)$$

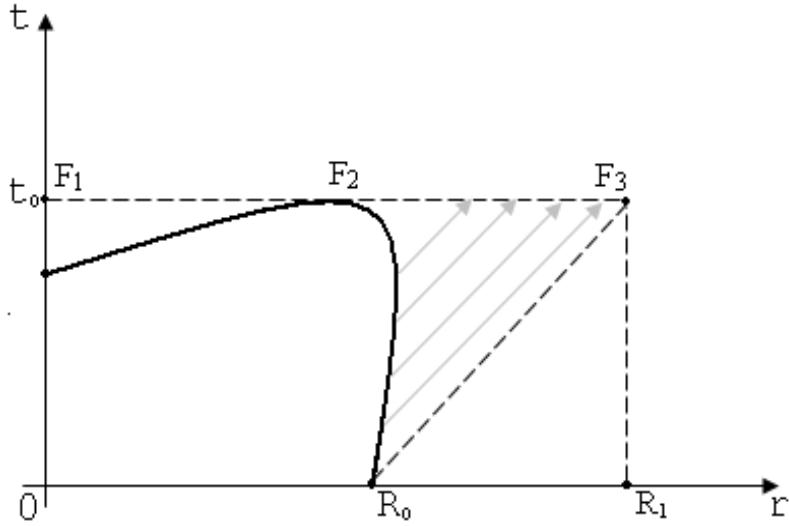


Figure 2: A sketch of the freeze-out hyper-surface (solid curve) for a spherically symmetric expansion of the fireball.

where $y^0 = \sigma(\mathbf{y})$ is the space-like hypersurface on which the wave function and its derivative are given.

Formula (47) gives a parametrization of the probability amplitude by the initial values of the wave function and its derivative at freeze-out times. The next steps are the same as in the section II B. The single-particle probability is obtained with a help of the density matrix: $P_1(\mathbf{p}) = \sum_{\gamma, \gamma'} \rho_{\gamma\gamma'} A_\gamma^*(\mathbf{p}) A_{\gamma'}(\mathbf{p}) = \sum_{\gamma, \gamma'} \rho_{\gamma\gamma'} \Psi_\gamma^{(+),*}(t_0, \mathbf{p}) \Psi_{\gamma'}^{(+)}(t_0, \mathbf{p})$. If we now represent the functions $\Psi_\gamma^{(+)}(t_0, \mathbf{p})$ as the Fourier integral

$$\Psi_\gamma^{(+)}(t_0, \mathbf{p}) = \int d^3x e^{-i\mathbf{p}\cdot\mathbf{x}} \Psi_\gamma^{(+)}(t_0, \mathbf{x}) = \int d^4x \delta(x^0 - t_0) e^{ip\cdot x} \Psi_\gamma^{(+)}(x), \quad (49)$$

and insert it to $P_1(\mathbf{p})$ we come to the standard expression of the single-particle probability $P_1(\mathbf{k}) = \int d^4X S(X, k)$, where we define the source function

$$S(X, K) = \delta(X^0 - t_0) \int d^4x \delta(x^0) e^{iK\cdot x} \sum_{\gamma, \gamma'} \rho_{\gamma\gamma'} \Psi_\gamma^{(+),*} \left(X + \frac{x}{2} \right) \Psi_{\gamma'}^{(+)} \left(X - \frac{x}{2} \right). \quad (50)$$

So, after definition the source function (50), which is constructed with the use of the wave function and its derivative at freeze-out times, we are ready to compare this parametrization of the source with the current one.

First of all let us mention that both parametrizations evidently coincide when one takes the special form of the current, $J_\gamma(t, \mathbf{x}) = \partial\Psi_\gamma(t, \mathbf{x})/\partial t \delta(t - t_0) + \Psi_\gamma(t_0, \mathbf{x}) \delta'(t - t_0)$, which

is expression on the r.h.s. of eq.(45). Then, we come to eq.(18) which is starting point in the current parametrization of the source. We show now that the same is valid in more general case (our consideration is very close to that one developed in the section II C).

The single-particle amplitude at asymptotic times (21) which we obtained in the section IIB can be written in the following way

$$\tilde{A}_\gamma(\mathbf{k}) = \lim_{x^0 \rightarrow \infty} \int d^3x \int d^4y \theta(x^0 - y^0) \left[f_k^{(+),*}(x) i \frac{\overleftrightarrow{\partial}}{\partial x^0} G^{(+)}(x - y) \right] J_\gamma(y), \quad (51)$$

where we use representation (B20) of the retarded Green's function, $G_R(x - y) = \theta(x^0 - y^0) [G^{(+)}(x - y) - G^{(-)}(x - y)]$, and orthogonality relation of the basic set of functions $f_k^{(+)}(x)$ and $f_k^{(-)}(x)$. To distinguish the “current” amplitude (51) from the “wave function” one (47) we marked it by tilde. On the next step we use the group property of the functions $G^{(+)}(x - y)$ (see (B22))

$$G^{(+)}(x - y) = \int d^4z \delta(z^0 - t_0) \left[G^{(+)}(x - z) \frac{\overleftrightarrow{\partial}}{\partial z^0} G^{(+)}(z - y) \right]. \quad (52)$$

Inserting this expression into (51) and making the following convolution

$$\int d^3x f_k^{(+),*}(x) i \frac{\overleftrightarrow{\partial}}{\partial x^0} G^{(+)}(x - z) = i f_k^{(+),*}(z), \quad (53)$$

we get the amplitude (51) in the form

$$\begin{aligned} \tilde{A}_\gamma(\mathbf{k}) &= \int d^4y d^4z \delta(z^0 - t_0) \left[f_k^{(+),*}(z) i \frac{\overleftrightarrow{\partial}}{\partial z^0} G^{(+)}(z - y) \right] J_\gamma(y) \\ &= \int d^4y d^4z \delta(z^0 - t_0) \left[f_k^{(+),*}(z) i \frac{\overleftrightarrow{\partial}}{\partial z^0} G_R(z - y) \right] J_\gamma(y), \end{aligned} \quad (54)$$

where the last line in (54) is obtained under assumption that the source current “works” just during the life time of the fireball, i.e. $J_\gamma(t, \mathbf{x}) \propto \theta(t_0 - t)$. Taking into account this feature one can define the wave function at freeze-out times as:

$$\Psi_\gamma(t_0, \mathbf{z}) = \int d^4y G_R(t_0 - y^0, \mathbf{z} - \mathbf{y}) J_\gamma(y^0, \mathbf{y}). \quad (55)$$

Inserting this notation to the second line on the r.h.s. of (54) one can rewrite the amplitude $\tilde{A}_\gamma(\mathbf{k})$ in the following way

$$\tilde{A}_\gamma(\mathbf{k}) = \int d^4z \delta(z^0 - t_0) \left[f_k^{(+),*}(z) i \frac{\overleftrightarrow{\partial}}{\partial z^0} \Psi_\gamma(z) \right], \quad (56)$$

Then, as will readily be observed the last expression coincide literally with the amplitude obtained in the wave function parametrization of the source (47). Hence, we can write

$$\tilde{A}_\gamma(\mathbf{k}) = A_\gamma(\mathbf{k}). \quad (57)$$

Because, the group property can be written in covariant form as well $G^{(+)}(x - y) = \int_{(\sigma)} d\sigma^\mu(z) G^{(+)}(x - z) (\vec{\partial} / \partial z^\mu) G^{(+)}(z - y)$, one can obtain the amplitude in the covariant form (48). Then, equality (57) is valid for an arbitrary space-like hyper-surface.

So, if we keep relation between current and wave function at freeze-out times in the form (55) we guarantee that the single-particle momentum amplitude will be the same in both approaches. The statement is valid also for two-particle momentum amplitude. Because the amplitude is the main constructive element of the single-particle probability (23) and two-particle probability (30), the equality of the amplitudes results in the equality of probabilities. This means that the source functions obtained in the wave function parametrization (50) and in the current parametrization of the source (4) are equal as well when eq.(55) is valid.

It is necessary to clarify the time structure of the current $J_\gamma(t, \mathbf{x})$. As a source of the single-particle state $\Psi_\gamma(t, \mathbf{x})$ the current acts during the life time of the fireball or when its time argument t is less than freeze-out times, $t \leq t_\sigma$:

$$J_\gamma(t, \mathbf{x}) \propto \theta(t_\sigma - |t|). \quad (58)$$

In the applications the cutting of the time interval is usually made in a soft way with a help of the Gaussian function, $J_\gamma(t, \mathbf{x}) \propto \exp(-t^2/2\tau^2)$, where τ is of the same order as t_σ . Our previous consideration was based on the rapid cutting of the current on the freeze-out hyper-surface like that in (58). Can a smooth switching off destroy our scheme? It is necessary to point out that just the Fourier transformed quantities enter the single-particle and two-particle probabilities. Let us look at the shape of the Fourier transformed cutting profiles. It is interesting to note that the Fourier components of the both time cutting functions, the Gaussian function and θ -function (58), give approximately the same bell like dependence on energy variable E , $J(E) = \int_{-\infty}^{\infty} dt J(t) \exp(iEt)$. These functions squared, $J(E)^2$, are depicted in Fig.3. Only a slight difference between these functions is seen and, therefore, the choice of the type of time cutting function does not affect much, at least qualitatively they give the same result. That is why, if we exploit the θ -function cutting rule (58) we obtain the same probabilities to register the particles as in the case of the Gaussian profile.

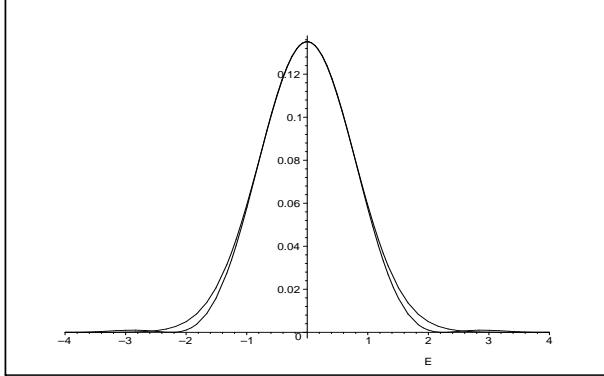


Figure 3: Comparison of the Fourier coefficients squared of the Gaussian and θ -function time cutting of a source current.

III. DISCUSSION AND CONCLUSIONS

We considered two types of a semi-classical parametrization of the source which give a transparent scheme of evaluation of the single-particle spectrum and two-particle correlations: the wave function and current parametrization of the source. The main ingredients of the wave function parametrization are the values of the wave function which are given on freeze-out hyper-surface (in relativistic approach the values of the wave function derivative should be given as well). In describing a propagation of the particles to detector after freeze-out these values serve as the initial conditions in the Cauchy problem: all information about evolution of the fireball is accumulated in the single-particle wave function, $\psi_\gamma(t_0, \mathbf{x})$, given at freeze-out times (see Fig.2). In relativistic case it is $\Psi_\gamma(t_0, \mathbf{x})$ and its derivative $\partial\Psi_\gamma(t_0, \mathbf{x})/\partial t$, then, the relativistic projection onto the out-state results that just the positive-energy defined part of the wave-function, $\Psi_\gamma^{(+)}(t_0, \mathbf{x})$, is exploited. For the sake of simplicity we discuss here a flat space-like hyper-surface $t = t_0 = \text{const}$. An arbitrary freeze-out hyper-surface is also considered in the paper.

Once the wave function at freeze-out times, $t = t_0$, is given, then, the single-particle spectrum and two-particle correlations can be constructed with the help of the single-particle Wigner density $S(X, K)$ which reads

$$S(X, K) = \delta(X^0 - t_0) \int d^4x \delta(x^0) e^{iK \cdot x} \sum_{\gamma, \gamma'} \rho_{\gamma\gamma'} \psi_\gamma^* \left(X + \frac{x}{2} \right) \psi_{\gamma'} \left(X - \frac{x}{2} \right), \quad (59)$$

where the measure of integration appears as a result of the transformation: $d^4x_1 d^4x_2 \delta(x_1^0 - t_0) \delta(x_2^0 - t_0) = d^4X d^4x \delta(X^0 - t_0) \delta(x^0)$ with $X = (x_1 + x_2)/2$, $x = (x_1 - x_2)$. To obtain the

source function in relativistic picture one should put in (59) the functions $\Psi_\gamma^{(+)}(x)$ in place of $\psi_\gamma(x)$, then, we come to expression (50).

We propose a scheme to generate the values of the wave function at freeze-out times. This can be done with a help of the current which parameterizes the source

$$\Psi_\gamma(t_0, \mathbf{x}) = \int dy^0 d^3y G_R(t_0 - y^0, \mathbf{x} - \mathbf{y}) J_\gamma(y^0, \mathbf{y}), \quad (60)$$

where $J_\gamma(y^0, \mathbf{y}) \propto \theta(t_0 - y^0)$, i.e. the life time of the current equals the life time of the fireball. If the wave function $\Psi_\gamma(t_0, \mathbf{x})$ and the current $J_\gamma(y^0, \mathbf{y})$ which parameterizes the source are in relation (60), then, the single- and two-particle spectra evaluated in the wave function parametrization are equal to the same quantities in the current parametrization. This results in equality of the source function (59) (or (50) in relativistic approach) obtained in the wave function parametrization of the source with the source function (4) obtained in the current parametrization. Moreover, the correspondence (60) results in extension of the space-like piece of the freeze-out hyper-surface by including the new piece which is created by the particles emitted from the fireball during its life, $0 \leq t \leq t_0$, through the time-like part of the freeze-out hyper-surface. As it is seen in Fig.1 the space-like part of the freeze-out hyper-surface, segment F_0F_1 , is extended by adding a new piece, segment F_1F_2 , which is created by the particles emitted from the boundary, segment F_1R_0 . The same picture takes the place for the freeze-out hyper-surface of an arbitrary shape which is sketched in Fig.2: the segment F_2F_3 accumulates the particles emitted from the time-like part of the freeze-out hyper-surface during time span t_0 . Hence, if the wave function at freeze-out times is generated by the source current as in (60), then, the wave function accumulates information about all particles emitted from the fireball. Moreover, the correspondence (60) (the correspondence (41) in nonrelativistic case), as it is seen in Figs.1, 2 results in extension of the size of the fireball: the wave function parametrization reflects the radius of the system which is R_1 (but not R_0). So, effectively the size of the system is bigger, an extended volume includes the spherical layer, $R_0 \leq r \leq R_1$, which contains free particles emitted from the boundary of the fireball.

However, there is a source of particles, for instance pions, which creates particles after freeze-out, for instance a decay of long lived resonances. It can be formalized by introducing of a “post freeze-out” current, $J_{\text{pfo}}(x)$. Basically, from the very beginning we can separate current into two parts: the first part is a current before freeze-out, $I_\gamma(x)$, and the second

part is a current after freeze-out, $I_{\text{pfo}}(x)$,

$$J_\gamma(t, \mathbf{x}) = \theta(t_0 - t) I_\gamma(t, \mathbf{x}) + \theta(t - t_0) I_{\text{pfo}}(t, \mathbf{x}).$$

Then, the momentum amplitude (56) should be modified to the following form

$$A_\gamma(\mathbf{p}) = \int d^4x \delta(x^0 - t_0) f_p^{(+),*}(x) i \overleftrightarrow{\frac{\partial}{\partial x^0}} \Psi_\gamma(x) + i \int d^4x \theta(x^0 - t_0) f_p^{(+),*}(x) I_{\text{pfo}}(x), \quad (61)$$

where the second term on the r.h.s. of equation reflects the sources of particles which appear after freeze-out, i.e. for times $t \geq t_0$. It turns out that these particles give contribution just into the single-particle spectrum. Because for the particles which are created after freeze-out a symmetrization, for instance of two-particle wave function, starts when the particles are separated by big distances the two-particle momentum probability, $P_2(K, q)$, has appreciable values just for small relative momenta, $|\mathbf{q}| \leq 10 \div 20 \text{ MeV}/c$. These values of the relative momenta are not experimentally “visible”.

Acknowledgements: The work of D. Anchishkin was partially supported by the CERN TH Division (The Scientific Associates programme) and by the program ”Fundamental properties of the physical systems under extreme conditions” of the Section for physics and astronomy of the National Academy of Sciences of Ukraine. The work of U. Heinz was supported by the U.S. Department of Energy under contract DE-FG02-01ER41190.

APPENDIX A: NONRELATIVISTIC APPROXIMATION OF THE KLEIN-GORDON EQUATION

In this appendix we are going to obtain a relation between the current which appears as a right part (source) of the Klein-Gordon equation and a current (source) of the Schrödinger equation. For this purpose we take eq. (18), which determines a current parametrization of the source, and consider it in a non-relativistic limit. This equation can be rewritten in the following way

$$(\partial_\mu \partial^\mu + m^2) \Psi_\gamma(x) = -(i\partial_t + \sqrt{m^2 - \nabla^2}) (i\partial_t - \sqrt{m^2 - \nabla^2}) \Psi_\gamma(t, \mathbf{x}) = J_\gamma(t, \mathbf{x}), \quad (A1)$$

where $t = x^0$. We make a standard unitary transformation of the wave function to extract oscillations associated with particle mass

$$\Psi_\gamma(x) = e^{-imx^0} \psi_\gamma(x). \quad (A2)$$

With respect to the new wave function $\psi_\gamma(x)$ eq. (A1) reads

$$(i\partial_t + m + \sqrt{m^2 - \nabla^2})(i\partial_t + m - \sqrt{m^2 - \nabla^2})\psi_\gamma(t, \mathbf{x}) = -e^{imt} J_\gamma(t, \mathbf{x}). \quad (\text{A3})$$

It is necessary to point out that from now on the energy operator $i\partial_t$ with respect to the wave function $\psi_\gamma(x)$ is an operator of the kinetic energy because $i\partial_t\psi_\gamma(t, \mathbf{x}) = e^{imt}(i\partial_t - m)\Psi_\gamma(t, \mathbf{x})$. We shall also quote our consideration to positive energies. That is why in the non-relativistic approximation we have the following inequalities for energy and momentum operators: $\langle i\hbar\partial_t \rangle / mc^2 \ll 1$ and $\langle -i\hbar\nabla \rangle / mc \ll 1$, where the broken brackets mean an averaging over some single-particle quantum state. Hence, the operator in the first bracket on the l.h.s. of eq. (A3) is a positive definite operator (it does not have zero eigenvalues). As a consequence, it always has inverse operator, that is why we write

$$(i\partial_t + m - \sqrt{m^2 - \nabla^2})\psi_\gamma(t, \mathbf{x}) = -\left(i\partial_t + m + \sqrt{m^2 - \nabla^2}\right)^{-1} e^{imt} J_\gamma(t, \mathbf{x}). \quad (\text{A4})$$

With making use of the relation $-i\nabla\psi_\gamma(t, \mathbf{x}) \ll m\psi_\gamma(t, \mathbf{x})$ one can expand square root operators which we meet on the l.h.s. and on the r.h.s. of this equation in the Taylor series. Just keeping leading terms we arrive to the non-relativistic equation

$$\left(i\partial_t + \frac{1}{2m}\nabla^2\right)\psi_\gamma(t, \mathbf{x}) = -\frac{1}{2m}e^{imt}J_\gamma(t, \mathbf{x}), \quad (\text{A5})$$

where we skipped all terms of the order $1/c^2$ and higher, they serve as relativistic corrections. A general scheme to obtain the non-relativistic equation of motion to any order of relativistic corrections from the relativistic equation in the presence of an external field was elaborated in [18].

Appendix B: INITIAL CONDITIONS AS AN EXTERNAL CURRENT

1. Cauchy problem for Schrödinger equation

In this section we represent the initial conditions of a differential equation as an external current (the generalized Cauchy problem [16]). We consider a differential equation which contains a first derivative with respect to time. To be specific let us take the Schrödinger equation

$$(i\partial_t - \hat{H}(\mathbf{x}))\psi(x) = 0 \quad (\text{B1})$$

with initial condition: $\psi(t = 0, \mathbf{x}) = \Phi(\mathbf{x})$. To solve eq. (B1) we define a new wave function $\bar{\psi}(t, \mathbf{x})$ by extension of $\psi(t, \mathbf{x})$ for $t < 0$ in the following way

$$\bar{\psi}(\mathbf{x}, t) = \begin{cases} \psi(\mathbf{x}, t), & \text{if } t \geq 0, \\ 0, & \text{if } t < 0, \end{cases}$$

which obviously satisfies eq. (B1) if $\psi(x)$ is a solution. As a next step we make the Fourier transformation of eq. (B1) separating the integral over time in two parts, $\int_{-\infty}^{\infty} dt F(t) = \int_{-\infty}^{0^-} dt F(t) + \int_0^{\infty} dt F(t)$. Then, (B1) reads

$$\begin{aligned} & \int_{-\infty}^{\infty} dt e^{i\omega t} i \partial_t \bar{\psi}(t, \mathbf{x}) - \hat{H}(\mathbf{x}) \int_{-\infty}^{\infty} dt e^{i\omega t} \bar{\psi}(t, \mathbf{x}) \\ &= i e^{i\omega t} \psi(t, \mathbf{x}) \Big|_0^{\infty} - i^2 \omega \int_0^{\infty} dt e^{i\omega t} \psi(t, \mathbf{x}) - \hat{H}(\mathbf{x}) \bar{\psi}(\omega, \mathbf{x}) = 0, \end{aligned} \quad (\text{B2})$$

where $\bar{\psi}(\omega, \mathbf{x}) = \int_{-\infty}^{\infty} dt e^{i\omega t} \bar{\psi}(t, \mathbf{x})$. After integration by parts one can insert the function $\Phi(\mathbf{x})$, which represents the initial condition, to eq. (B2) keeping in mind that $e^{i\omega t} \psi(t, \mathbf{x}) \Big|_{t \rightarrow \infty} \rightarrow 0$. Then, for the Fourier components we obtain equation

$$(\omega - \hat{H}(\mathbf{x})) \bar{\psi}(\omega, \mathbf{x}) = i \Phi(\mathbf{x}). \quad (\text{B3})$$

Let us make the inverse Fourier transformation of eq. (B3) with making use of the equality:

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \omega \bar{\psi}(\omega, \mathbf{x}) = i \partial_t \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \bar{\psi}(\omega, \mathbf{x}) = i \partial_t \bar{\psi}(t, \mathbf{x}).$$

Then, one obtains equation in time-space representation

$$(i \partial_t - \hat{H}(\mathbf{x})) \bar{\psi}(t, \mathbf{x}) = i \Phi(\mathbf{x}) \delta(t). \quad (\text{B4})$$

The same equation is valid for the function $\psi(t, \mathbf{x})$. It is an interesting result because we reduce the initial condition for differential equation (B1) to impulse current which stands now on the r.h.s. of equation (B4).

One can define the Green's function $(i \partial_t - \hat{H}(\mathbf{x})) G(x - y) = \delta^4(x - y)$. With a help of the Green's function one can write solution of eq. (B4)

$$\psi(t, \mathbf{x}) = i \int d^3y G(t - t_0, \mathbf{x} - \mathbf{y}) \Phi(\mathbf{y}). \quad (\text{B5})$$

On the other hand, it is solution of the Cauchy problem which was formulated as eq. (B1) with initial condition. Integral on the r.h.s. of (B5) represents propagation of the initial

”excitation” $\Phi(\mathbf{y})$ which exists at time $t = t_0$ to other spatial points \mathbf{x} during time interval $(t - t_0)$.

In free case in the non-relativistic limit the Green’s function $G_0(x - y)$ satisfies equation $(i\partial_t + \nabla^2/2m) G_0(x - y) = \delta^4(x - y)$, and it reads

$$G_0(x) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik\cdot x}}{k^0 - \omega(\mathbf{k}) + i\epsilon} = -i \theta(x^0) \left(\frac{m}{2\pi i x^0} \right)^{3/2} \exp \left[i \frac{m\mathbf{x}^2}{2x^0} \right], \quad (\text{B6})$$

where $\omega(\mathbf{k}) = \mathbf{k}^2/2m$. Note, the non-relativistic Green’s functions have the following group property

$$\int d^3y G_0(x - y) G_0(y - z) = -i G_0(x - z). \quad (\text{B7})$$

So, using the explicit expression of the free Green’s function (B6) one can write solution of the Cauchy problem accumulated in eq. (B4) in the following form

$$\begin{aligned} \psi(t, \mathbf{x}) &= \theta(t) \int \frac{d^3k}{(2\pi)^3} d^3y e^{-i\omega(\mathbf{k})t + i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \Phi(\mathbf{y}) = \theta(t) \int \frac{d^3k}{(2\pi)^3} e^{-i\omega(\mathbf{k})t + i\mathbf{k}\cdot\mathbf{x}} \Phi(\mathbf{k}) \\ &= \theta(t) \left(\frac{m}{2\pi i t} \right)^{3/2} \int d^3y \exp \left[i \frac{m(\mathbf{x} - \mathbf{y})^2}{2t} \right] \Phi(\mathbf{y}), \end{aligned} \quad (\text{B8})$$

where $\Phi(\mathbf{k}) = \int d^3y \exp(-i\mathbf{k}\cdot\mathbf{y}) \Phi(\mathbf{y})$.

We turn now to another way of solution of the Cauchy problem (B1). Formally one can write the solution of the problem as

$$\psi(t, \mathbf{x}) = \theta(t - t_0) e^{-i\hat{H}(\hat{\mathbf{p}}, \mathbf{x})(t-t_0)} \Phi(\mathbf{x}), \quad (\text{B9})$$

where $\hat{\mathbf{p}} = -i\hbar\partial/\partial\mathbf{x}$ is the momentum operator and $\psi(t = t_0, \mathbf{x}) = \Phi(\mathbf{y})$, in what follows we adopt $t_0 = 0$. We are going to give the formal solution (B9) in coordinate representation and then compare it with (B8). Indeed, in coordinate representation (B9) reads

$$\psi(t, \mathbf{x}) = \theta(t) \int \frac{d^3p}{(2\pi)^3} \frac{d^3p'}{(2\pi)^3} d^3x' \langle \mathbf{x} | \mathbf{p} \rangle \langle \mathbf{p} | e^{-i\hat{H}(\hat{\mathbf{p}}, \mathbf{x})t} | \mathbf{p}' \rangle \langle \mathbf{p}' | \mathbf{x}' \rangle \Phi(\mathbf{x}'), \quad (\text{B10})$$

where $\langle \mathbf{x} | \mathbf{p} \rangle = \exp(i\mathbf{p}\cdot\mathbf{x})$ is the eigen function of the momentum operator. In free case the matrix elements of the Hamiltonian, $\hat{H}_0(\hat{\mathbf{p}}, \mathbf{x}) = \hat{\mathbf{p}}^2/2m$, looks like, $\langle \mathbf{p} | e^{-i\hat{H}_0(\hat{\mathbf{p}}, \mathbf{x})t} | \mathbf{p}' \rangle = (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{p}') e^{-i\omega(\mathbf{p})t}$, where $\omega(\mathbf{p}) = \mathbf{p}^2/2m$. Finally, for free case we can write (B9) in coordinate representation

$$\psi(t, \mathbf{x}) = \theta(t) \int \frac{d^3p}{(2\pi)^3} d^3x' e^{-i\omega(\mathbf{p})t + i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')} \Phi(\mathbf{x}'). \quad (\text{B11})$$

We see that this expression coincides with (B8). So, we find that it does not matter in what approach one solves the Cauchy problem, with the help of the Green’s function or using evolution operator, both approaches give the same result.

2. Cauchy problem for relativistic equation

We consider the case of free scalar field which can be described by the Klein-Gordon equation

$$(\partial_\mu \partial^\mu + m^2) \Psi(x) = 0, \quad (\text{B12})$$

which is supplemented by the initial conditions

$$\Psi(t=0, \mathbf{x}) = \Phi_0(\mathbf{x}), \quad \text{and} \quad \frac{\partial \Psi(t=0, \mathbf{x})}{\partial t} = \Phi_1(\mathbf{x}). \quad (\text{B13})$$

We are going to show how these initial conditions can be inserted into eq. (B12) as a specific current. As the first step let us extend the function $\Psi(t, \mathbf{x})$ (we define the function equals to zero for negative times)

$$\bar{\Psi}(\mathbf{x}, t) = \begin{cases} \Psi(\mathbf{x}, t), & \text{if } t \geq 0, \\ 0, & \text{if } t < 0. \end{cases}$$

We make the Fourier transformation of eq. (B12) separating the integral over time in two parts, $\int_{-\infty}^{\infty} dt F(t) = \int_{-\infty}^{0^-} dt F(t) + \int_0^{\infty} dt F(t)$. Then, one obtains

$$\begin{aligned} \int_{-\infty}^{\infty} dt e^{i\omega t} (\partial_\mu \partial^\mu + m^2) \Psi(x) &= \int_{-\infty}^{\infty} dt e^{i\omega t} \partial_t^2 \bar{\Psi}(t, \mathbf{x}) + (-\partial_{\mathbf{x}}^2 + m^2) \int_{-\infty}^{\infty} dt e^{i\omega t} \bar{\Psi}(t, \mathbf{x}) \\ &= e^{i\omega t} \frac{\partial \bar{\Psi}(t, \mathbf{x})}{\partial t} \Big|_0^\infty - i\omega \int_0^{\infty} dt e^{i\omega t} \frac{\partial \bar{\Psi}(t, \mathbf{x})}{\partial t} + (-\partial_{\mathbf{x}}^2 + m^2) \bar{\Psi}(\omega, \mathbf{x}) \\ &= -\Phi_1(\mathbf{x}) + i\omega \Phi_0(\mathbf{x}) - \omega^2 \bar{\Psi}(\omega, \mathbf{x}) + (-\partial_{\mathbf{x}}^2 + m^2) \bar{\Psi}(\omega, \mathbf{x}), \end{aligned} \quad (\text{B14})$$

where $\bar{\Psi}(\omega, \mathbf{x}) = \int_{-\infty}^{\infty} dt e^{i\omega t} \bar{\Psi}(t, \mathbf{x})$. After two integrations by parts we insert the functions $\Phi_0(\mathbf{x})$ and $\Phi_1(\mathbf{x})$, which represent the initial conditions, to eq. (B14) keeping in mind that $e^{i\omega t} \bar{\Psi}(t, \mathbf{x}) \Big|_{t \rightarrow \infty} \rightarrow 0$. Then, for the Fourier components we obtain the following equation

$$(-\omega^2 - \partial_{\mathbf{x}}^2 + m^2) \bar{\Psi}(\omega, \mathbf{x}) = \Phi_1(\mathbf{x}) - i\omega \Phi_0(\mathbf{x}). \quad (\text{B15})$$

We make the inverse Fourier transformation of eq. (B15) with making use of the equalities:

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} (-\omega^2) \bar{\Psi}(\omega, \mathbf{x}) = \partial_t^2 \bar{\Psi}(t, \mathbf{x}), \quad \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} (-i\omega) e^{-i\omega t} = \frac{d}{dt} \delta(t).$$

Then, one obtains equation in the space-time representation

$$(\partial_t^2 - \partial_{\mathbf{x}}^2 + m^2) \bar{\Psi}(t, \mathbf{x}) = \Phi_1(\mathbf{x}) \delta(t) + \Phi_0(\mathbf{x}) \frac{d}{dt} \delta(t). \quad (\text{B16})$$

The same equation is valid for the function $\Psi(t, \mathbf{x})$ for times, $t \geq 0$,

$$(\partial_\mu \partial^\mu + m^2) \Psi(x) = \Phi_1(\mathbf{x}) \delta(t) + \Phi_0(\mathbf{x}) \delta'(t). \quad (\text{B17})$$

So, we represent the initial conditions for the differential equation (B12) as a current which stands on the r.h.s. of equation (B17). The use of the function $\delta'(t)$ is common: $\int dt F(t) \delta'(t) = - \int dt \frac{dF(t)}{dt} \delta(t)$.

To solve eq.(B17) we use the Green's functions $G_R(x - y)$ (or $G_F(x - y)$). Because in further consideration we look for solutions $\Psi(x^0, \mathbf{x})$ at asymptotic times, $x^0 \rightarrow \infty$, just a piece of the propagator $G_R(x - y)$ which carries the positive defined frequencies really gives contribution. (Because of that it does not matter what kind of the Green's function should be used, retarded or causal one.) So, with the help of the propagator $G_R(x - y)$ which is defined as, $(\partial_\mu \partial^\mu + m^2) G_R(x - y) = \delta^4(x - y)$, we can write solution of eq. (B17)

$$\begin{aligned} \Psi(x) &= \int d^4y G_R(x - y) [\Phi_1(\mathbf{y}) \delta(y^0) + \Phi_0(\mathbf{y}) \delta'(y^0)] \\ &= \int d^4y \delta(y^0) \left[G_R(x - y) \frac{\partial \Psi(y^0, \mathbf{y})}{\partial y^0} - \frac{\partial G_R(x - y)}{\partial y^0} \Psi(y^0, \mathbf{y}) \right]. \end{aligned} \quad (\text{B18})$$

The integration in (B18) is going on the space-like hypersurface $y^0 = 0$. For arbitrary hypersurface σ it can be written in the covariant form in the following way (see, for instance, [20], ch. 7)

$$\Psi(x) = \int_{(\sigma)} d\sigma^\mu(y) G_R(x - y) \overleftrightarrow{\frac{\partial}{\partial y^\mu}} \Psi(y) \quad (\text{B19})$$

with $\sigma^\mu(y)$ as the space-like hypersurface on which the initial conditions $\Psi(y)$ and $\partial\Psi(y)/\partial y^\mu$ are given, i.e. these functions under the integral are defined when $y \in \sigma$. By definition, $f_1(t) \overleftrightarrow{\partial_t} f_2(t) \equiv f_1(t)\partial_t f_2(t) - \partial_t(f_1(t))f_2(t)$.

We derive now one more relation which is useful in the consideration. One can write the Green's functions in the following way

$$G_R(x - y) = \theta(x^0 - y^0) [G^{(+)}(x - y) - G^{(-)}(x - y)]. \quad (\text{B20})$$

where

$$G^{(\pm)}(x - y) = i \int \frac{d^3k}{(2\pi)^3 2\omega(\mathbf{k})} f_k^{(\pm)}(x) f_k^{(\pm),*}(y) \quad (\text{B21})$$

with $f_k^{(\pm)}(x) = e^{\mp k \cdot x}$, which obey the orthogonal relations. It is obvious that the functions $G^{(+)}(x - y)$ and $G^{(-)}(x - y)$ satisfy the Klein-Gordon equation, $(\partial_\mu \partial^\mu + m^2) G^{(\pm)}(x - y) = 0$.

With taking into account normalization, $\int d^3x f_k^{(+),*}(x) i \overleftrightarrow{\partial}_{x^0} f_p^{(+)}(x) = (2\pi)^3 2\omega(\mathbf{k}) \delta^3(\mathbf{k}-\mathbf{p})$, one obtains

$$\int d^4z \delta(z^0) \left[G^{(+)}(x-z) \frac{\partial G^{(+)}(z-y)}{\partial z^0} - \frac{\partial G^{(+)}(x-z)}{\partial z^0} G^{(+)}(z-y) \right] = G^{(+)}(x-y). \quad (\text{B22})$$

This can be written for arbitrary space-like hypersurface σ in the covariant form [20]

$$G^{(+)}(x-y) = \int_{(\sigma)} d\sigma^\mu(z) G^{(+)}(x-z) \frac{\overleftrightarrow{\partial}}{\partial z^\mu} G^{(+)}(z-y). \quad (\text{B23})$$

-
- [1] M. Gyulassy, S.K. Kauffmann and L.W. Wilson, Phys. Rev. C **20**, 2267 (1979).
- [2] D.H. Boal, C.-K. Gelbke and B.K. Jennings, Rev. Mod. Phys. **62**, 553 (1990).
- [3] U. Heinz and B.V. Jacak, Ann. Rev. Nucl. Part. Sci. **49**, 529 (1999); arXiv:nucl-th/9902020.
- [4] R. M. Weiner, Phys. Rept. **327**, 249 (2000).
- [5] Sandra S. Padula, Braz. J. Phys. **35**, 70 (2005); arXiv:nucl-th/0412103.
- [6] M. A. Lisa, S. Pratt, R. Soltz, U. Wiedemann, Ann. Rev. Nucl. Part. Sci. **55**, 357 (2005); arXiv:nucl-ex/0505014.
- [7] T. Csörgő, J. Phys. Conf. Ser. **50**, 259 (2006); arXiv:nucl-th/0505019.
- [8] D. Anchishkin, U. Heinz, and P. Renk, Phys. Rev. C **57**, 1428 (1998); arXiv:nucl-th/9710051.
- [9] STAR Collaboration, C. Adler et al., Phys. Rev. Lett. **87**, 082301 (2001).
- [10] PHENIX Collaboration, K. Adcox et al., Phys. Rev. Lett. **88**, 192302 (2002).
- [11] PHENIX Collaboration, S. S. Adler et al., Phys. Rev. Lett. **93**, 152302 (2004).
- [12] STAR Collaboration, J. Adams et al., Phys. Rev. C**71**, 044906 (2005).
- [13] E. Frodermann, U. Heinz, and M. A. Lisa, Phys. Rev. C**73**, 044908 (2006); arXiv:nucl-th/0602023.
- [14] Q. Li, M. Bleicher, H. Stöcker, arXiv:0709.1409.
- [15] S. Chapman, and U. Heinz, Phys. Lett. B **340**, 250 (1994).
- [16] V.S. Vladimirov, Equations of Mathematical Physics, *Central Books Ltd, 1985*.
- [17] D.V. Anchishkin, W.A. Zajc and G.M. Zinovjev, Ukrainian J. Phys. **47**, 451 (2002); arXiv:nucl-th/9904061.
- [18] D. Anchishkin, J. Phys. A: Math. Gen. **30**, 1303 (1997).
- [19] S. Pratt, T. Csörgő, and J. Zimányi, Phys. Rev. C **42**, 2646 (1990).
- [20] Silvan S. Schweber, An Introduction to Relativistic Quantum Field Theory, *Row, Peterson and Co., N.Y., 1961*.